CTMCs

Quantitative Logics

17 April 2013

David N. Jansen

Transient state distribution

- Given initial distribution π_0,
  what is the probability distribution π_t
  over the states at time *t*?
- Differentiate $\pi_t = \pi_0 \mathbf{P}(t)$:
- $\pi_t' = \pi_0 \mathbf{P}'(t) = \pi_0 \mathbf{P}(t) \mathbf{Q} = \pi_t \mathbf{Q}$
- Differential equation with solution:
  $\pi_t = \pi_0 e^{\mathbf{Q}t} = \pi_0 \Sigma_{i=0\ldots\infty} (\mathbf{Q}t)^i / i!$

Steady-state distribution

- In steady-state, the distribution does not change any more, i.e. $\pi' = 0$
- but $\pi' = \pi \mathbf{Q}$

- Solve equation system: $0 = \pi \mathbf{Q}$
  (and $\Sigma_s \pi(s) = 1$)

How to calculate the exponential of a matrix

- $\pi_t = e^{\mathbf{Q}t} = \Sigma_{i=0\ldots\infty} (\mathbf{Q}t)^i / i!$ is numerically unstable

- transform sum using uniformisation!

Uniform DTMC of a CTMC

- Exit rate of a state := sum of all rates
- A CTMC is uniform if every state has the same exit rate.
- If a CTMC is uniform, the probability to get a certain number of transitions does not depend on the states visited.

Uniform DTMC of a CTMC

- Split CTMC into two parts:
  - timing: state sojourn times have cdf $F_E(t) = 1 - e^{-Et}$
  - states visited: probability of transitions is described by DTMC with $\mathbf{P} := \mathbf{R} / E = \mathbf{Q} / E + \mathbf{I}$

- A non-uniform CTMC can be uniformised (without changing π_t) by adding self-loops.

How to calculate the exponential of a uniform CTMC matrix

- Let $\mathbf{P} := \mathbf{Q} / E + \mathbf{I}$, so $\mathbf{Q} = E(\mathbf{P} - \mathbf{I})$
  where E = the maximum exit rate

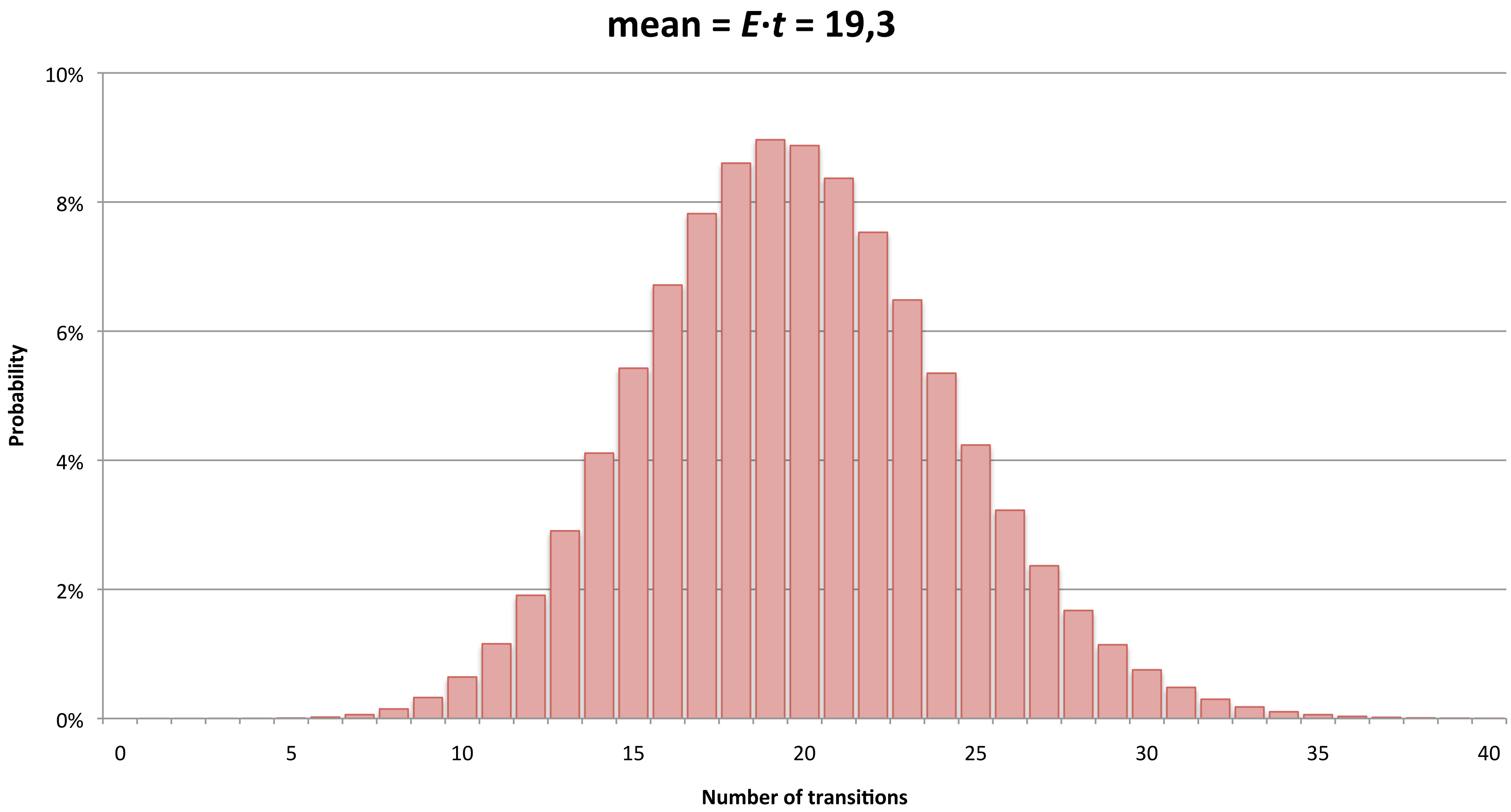
- $\pi_0 \, e^{\mathbf{Q}t}$
  $= \pi_0 \, \Sigma_{i=0\ldots\infty} \, (\mathbf{Q}t)^i / i!$
  $= \pi_0 \, \Sigma_{i=0\ldots\infty} \, (E[\mathbf{P} - \mathbf{I}]t)^i / i!$
  $= \pi_0 \, \Sigma_{i=0\ldots\infty} \, \Sigma_{j=0\ldots i} \, (E\mathbf{P}t)^j \, (-E\mathbf{I}t)^{i-j} \, i! \, / \, [j! \, (i-j)!] \, / \, i!$
  $= \pi_0 \, \Sigma_{j=0\ldots\infty} \, (E\mathbf{P}t)^j \, / \, j! \times \Sigma_{i=j\ldots\infty} \, (-Et)^{i-j} \, / \, (i-j)!$
  $= \pi_0 \, \Sigma_{j=0\ldots\infty} \, \mathbf{P}^j \, (Et)^j \, / \, j! \times e^{-Et}$

How many transitions are taken?

- Poisson distribution:
  $poi(i)$ = Prob(# of transitions in time $t = i$)
  $= e^{-Et} (Et)^i / i!$

- $\pi_0 \, e^{\mathbf{Q}t} = \pi_0 \sum_{j=0\ldots\infty} \mathbf{P}^j \, poi(j)$

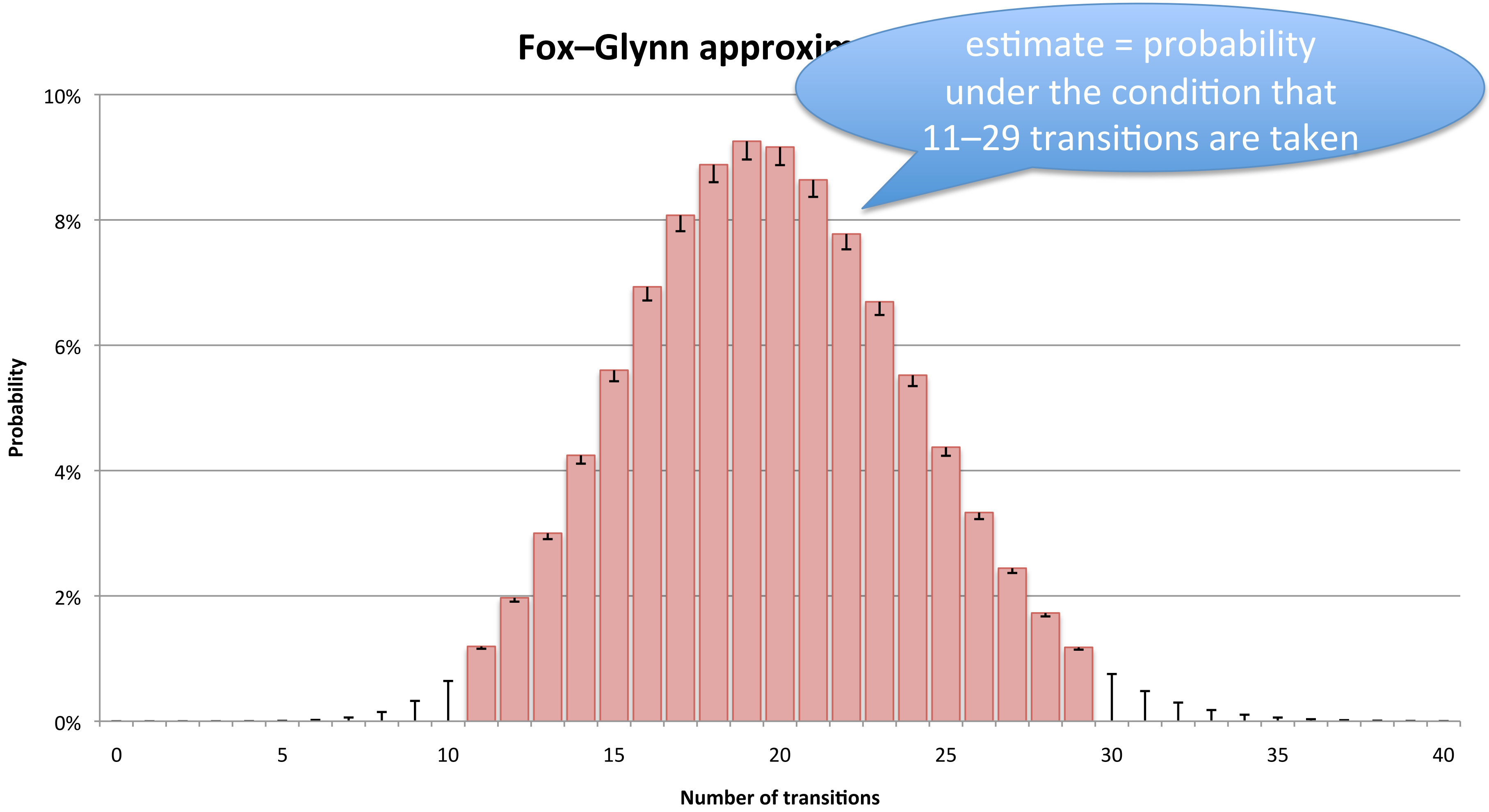
Poisson distribution



How to approximate the infinite sum

- # of transitions is most likely close to the mean
- Approximate *poi*(*i*) by probability under the condition that # of transitions ≈ mean
- Fox–Glynn algorithm finds required # of summands

Approximating this distribution



How to calculate $e^{\mathbf{Q}t}$

1. Uniformise the CTMC
2. Take the embedded DTMC
3. sum over the number of transitions, according to Fox–Glynn

CSL

- state formulas ϕ, ψ
  - a — atomic proposition
  - $\neg\phi$ — negation
  - $\phi \vee \psi$ — disjunction
  - $\mathbf{P}_{\leq p}(\Pi)$, $\mathbf{P}_{\geq p}(\Pi)$ — probabilistic operator
  - $\mathbf{S}_{\leq p}(\phi)$, $\mathbf{S}_{\geq p}(\phi)$ — steady-state operator
- path formulas Π
  - $X^I \phi$ — next state
  - $\phi\ U^I \psi$ — until

with time bound:
an interval $I \subseteq \mathbb{R}_{\geq 0}$

Examples:
Please translate to a formula

- Is the probability that the Hubble space telescope crashes within 10 years > 0.01?
- Is the probability that the NASA has to send two repair missions within 3 years ≤ 0.1?
- Will the Hubble space telescope crash?

Semantics

- path formulas: similar to PCTL
- (time-bounded) next: $X^I \phi$
  paths that take the first step within a time $\in I$
  and go to a ϕ-state
- (time-bounded) until: $\phi\ U^I\ \psi$
  paths that reach a ψ-state within a time in $\in I$
  and only pass via ϕ-states before

Measurability

- require that path sets be measurable
- definition of cylinder set for CTMC...
- path sets satisfying $X^I\ \phi$ or $\phi\ U^I\ \psi$ are unions of cylinder sets

Multiple until operator

- original definition included operator

$$\phi_1 \, U^{I_1} \, \phi_2 \, U^{I_2} \ldots U^{I_k} \, \phi_k$$

[Aziz/Sanwal/Singhal/Brayton: Verifying continuous time Markov chains. In: CAV 1996. LNCS 1102.] (contains errors)

- often reduced to k=2 only

[Baier/Haverkort/Hermanns/Katoen: Model-checking algorithms for continuous-time Markov chains. IEEE Trans. Softw. Eng. 2003.]

[Zhang/Jansen/Nielson/Hermanns: Efficient CSL model checking using stratification. Logical Methods Comp. Sci., 2012.]