CTMCs

Quantitative Logics

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Let’s play a game

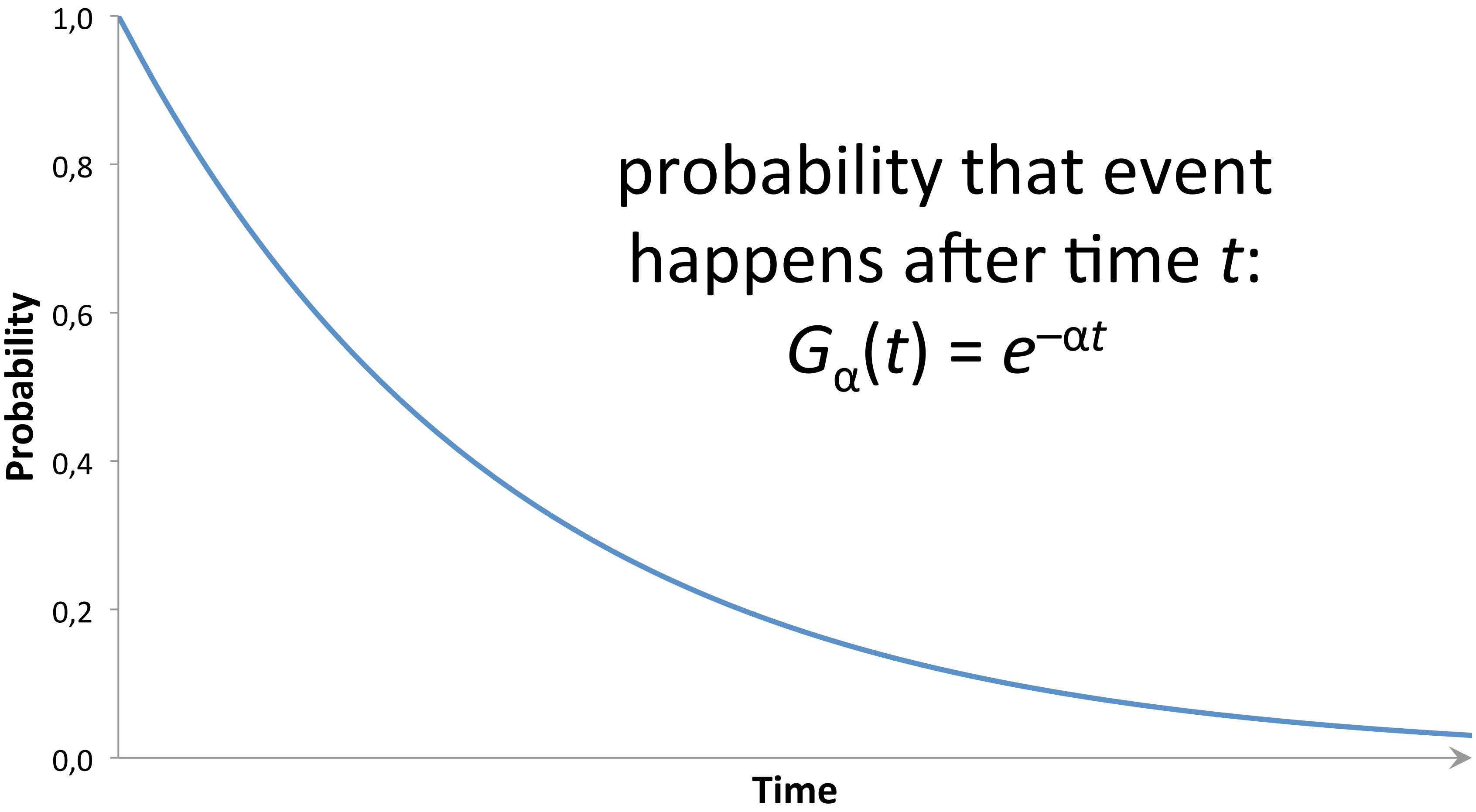
Recapitulation: DTMC

- A Markov chain consists of:
  - S — finite set of states (often $S = \{1, 2, \dots n\}$)
  - $\mathbf{P}: S \times S \rightarrow [0,1]$ — transition probability matrix (with row sums = 1)
  - $\pi_0: S \rightarrow [0,1]$ — initial state distribution (sometimes)
  - $L: S \rightarrow 2^{AP}$ — labelling with atomic propositions

what about time?

- Two interpretations of DTMC:
  - either every step takes the same time
  - or it is unknown / irrelevant how long a step takes
- Other timing models?

exponential distribution



exponential distribution

- cumulative distribution function (cdf) = probability that event happens ≤ time t:
  $F_\alpha(t) = 1 - G_\alpha(t) = 1 - e^{-\alpha t}$
- probability density function = derivative of cdf
  $f_\alpha(t) = F'_\alpha(t) = \alpha e^{-\alpha t}$
- expected value: $E(F_\alpha) = 1 / \alpha$

exponential distribution is memoryless

$$\Pr(\text{Event} > s+t \mid \text{Event} > s) =$$

$$\Pr(\text{Event} > s+t \wedge \text{Event} > s) / \Pr(\text{Event} > s) =$$

$$G(s+t) / G(s) =$$

$$e^{-\alpha(s+t)} / e^{-\alpha s} =$$

$$e^{-\alpha(s+t-s)} =$$

$$e^{-\alpha t} =$$

$$G(t) =$$

$$\Pr(\text{Event} > t)$$

minimum of exponential distributions

$$\Pr(\min(\text{Event}_\alpha, \text{Event}_\beta) > t) =$$
$$\Pr(\text{Event}_\alpha > t \wedge \text{Event}_\beta > t) =$$
$$G_\alpha(t) \cdot G_\beta(t) =$$
$$e^{-\alpha t} \cdot e^{-\beta t} =$$
$$e^{-(\alpha+\beta)t} =$$
$$G_{\alpha+\beta}(t) =$$
$$\Pr(\text{Event}_{\alpha+\beta} > t)$$

Idea for timing in Markov chains

- each transition from current state
  becomes enabled after a random time
  - with exponential distribution F_α
- first enabled transition is taken
  - intuition:
    Upon entering state, set random kitchen timers;
    first timer to ring indicates which transition is taken.
  - distribution of sojourn times
    = minimum of exponential distributions
  - probability to choose a transition
    = proportional to α

CTMC

- A continuous-time Markov chain consists of:
  - S — finite set of states (often $S = \{1, 2, \ldots n\}$)
  - $\mathbf{R}: S \times S \rightarrow \mathbb{R}_{\geq 0}$ — transition rate matrix
  - $\pi_0: S \rightarrow [0,1]$ — initial state distribution (sometimes)
  - $L: S \rightarrow AP$ — labelling with atomic propositions

Example

- Hubble space telescope

Probability space of a CTMC

- timed cylinder set
  $Cyl(s_0, I_1, s_1, \ldots, I_n, s_n)$:= paths starting with $s_0, s_1, \ldots, s_n$
  and with sojourn times in intervals $I_1, \ldots, I_n$
- σ-algebra generated by timed cylinder sets
- probability measure: unique extension of

$$\mu(Cyl(s_0, I_1, s_1, \ldots, I_n, s_n)) =$$
$$= \pi_0(s_0)\cdot \mathrm{Pr}(\mathrm{Event}_1 \in I_1)\cdot \mathbf{P}(s_0,s_1)\cdot$$
$$\mathrm{Pr}(\mathrm{Event}_2 \in I_2)\cdot \mathbf{P}(s_1,s_2)\cdots$$
$$\mathrm{Pr}(\mathrm{Event}_n \in I_n)\cdot \mathbf{P}(s_{n-1},s_n)$$

probability of sojourn time?

- $\Pr(\text{Event} \in (a,b]) =$
  $\Pr(\text{Event} > \text{a}) - \Pr(\text{Event} > \text{b}) = G(a) - G(b)$
- $\Pr(\text{Event} \in [a,b]) =$
  $\Pr(\text{Event} = a) + \Pr(\text{Event} \in (a,b]) =$
  $0 \qquad + G(a) - G(b)$
- concretely,
  $\Pr(\text{Event}_1 \in I_1) = G_1(\inf I_1) - G_1(\sup I_1) =$
  $e^{-\alpha \inf I_1} - e^{-\alpha \sup I_1}$
  where α = exit rate of $s_0 = \Sigma\, R(s_0, \ldots)$

rate of minimum of all exponential distributions

Infinitesimal generator Q

- matrix related to rate matrix
- used for state probabilities

- Assume $\mathbf{P}(t)$ = probability to move from ... to ... in time t
- $\mathbf{Q} := \mathbf{P}'(0)$

Properties of Q

- $q_{ij} = \lim_{dt \to 0} p_{ij}(dt) / dt$ (if $i \neq j$)
- $q_{ij}\, dt = p_{ij}(dt)$
  ≈ probability to move from i to j in time dt
  ≈ (rate to move from i to j) · $dt = \mathbf{R}(i,j)dt$
- $q_{ii} = \lim_{dt \to 0} (p_{ii}(dt) - 1) / dt$
- $-q_{ii}\, dt = 1 - p_{ii}(dt)$
  ≈ probability to leave i in time dt
  ≈ (exit rate of i) · $dt = \Sigma_{j \neq i}\, \mathbf{R}(i,j)dt$

Properties of Q

- $\mathbf{P}'(t) = \lim_{dt \to 0} [\, \mathbf{P}(t + dt) - \mathbf{P}(t)\,] \,/\, dt$

  $= \lim_{dt \to 0} [\, \mathbf{P}(t)\, \mathbf{P}(dt) - \mathbf{P}(t)\, \mathbf{I}\,] \,/\, dt$

  $= \mathbf{P}(t) \lim_{dt \to 0} [\, \mathbf{P}(dt) - \mathbf{I}\,] \,/\, dt$

  $= \mathbf{P}(t)\, \mathbf{Q}$