Probabilistic CTL

David N. Jansen

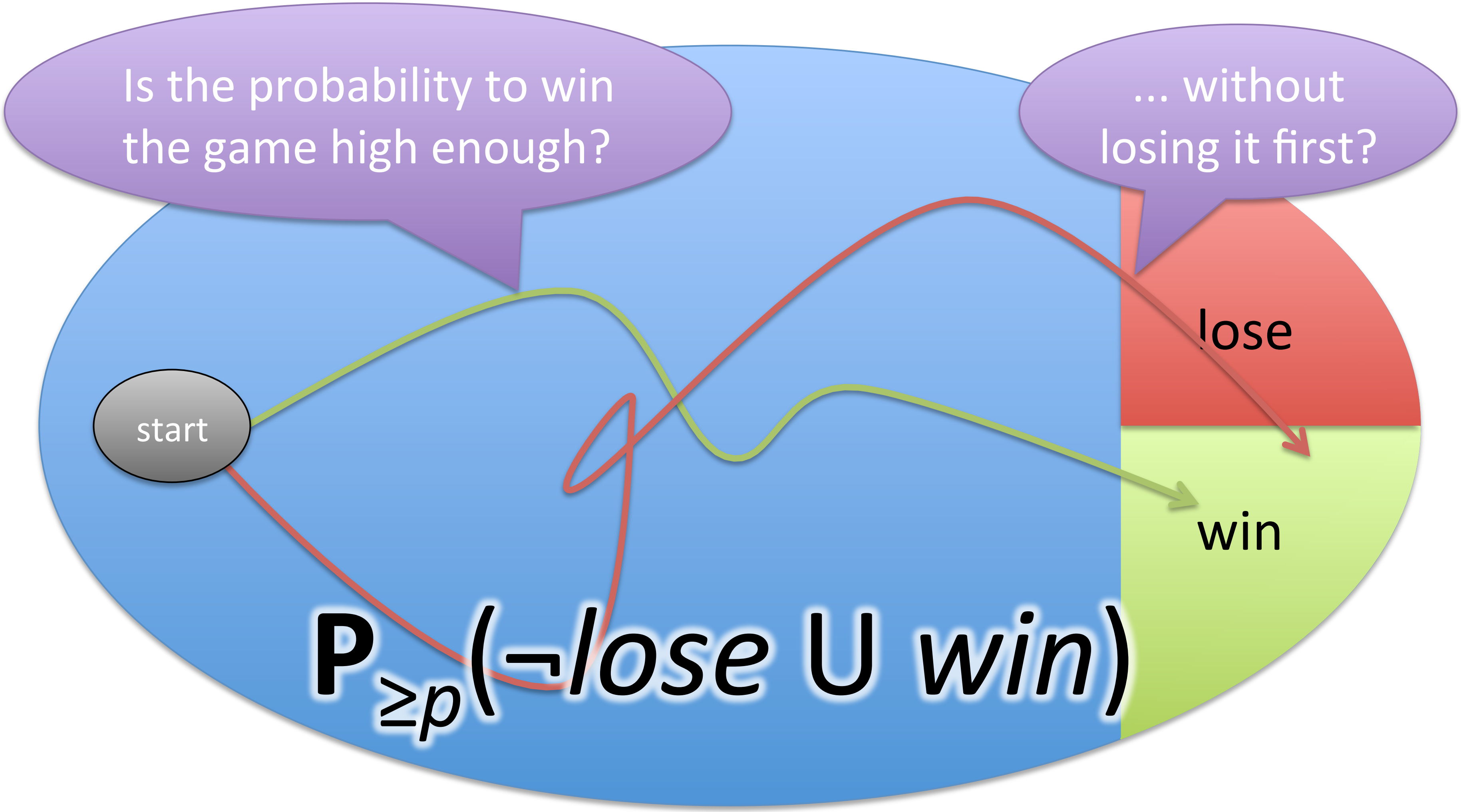
Recapitulation: DTMCs

- Markov chains describe
  the behaviour of discrete-state systems.
- Discrete-time Markov chains
  count the number of steps,
  but not how long a step takes.
- Transient state and steady-state analysis
  serve to calculate state probabilities.

Probabilistic CTL

- a logic to describe properties
  of Markov chains
- extends CTL
- strictly speaking, not a quantitative logic
  (truth values are Boolean: true or false)

Constrained Probabilistic Reachability



PCTL syntax

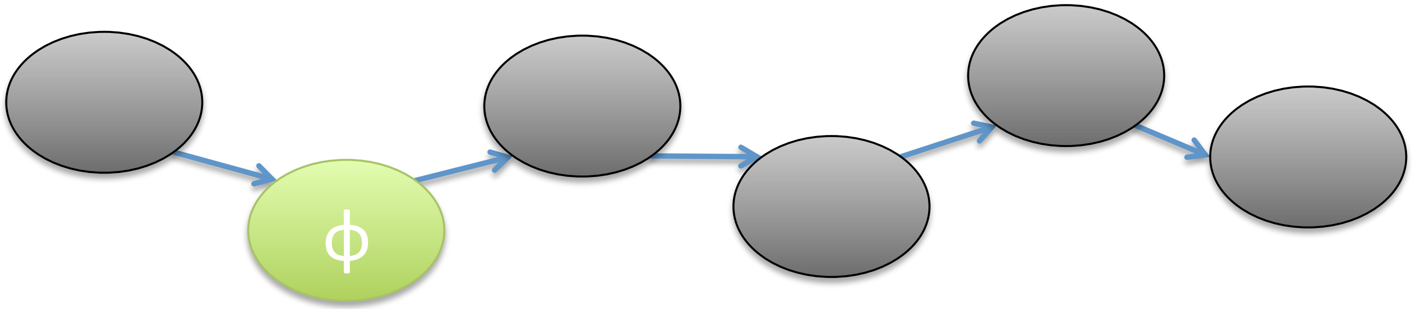
- state formulas ϕ, ψ
  - a — atomic proposition
  - $\neg\phi$ — negation
  - $\phi \vee \psi$ — disjunction
  - $\mathbf{P}_{<p}(\Pi)$, $\mathbf{P}_{\leq p}(\Pi)$, $\mathbf{P}_{\geq p}(\Pi)$, $\mathbf{P}_{>p}(\Pi)$ — probability constraint

$p \in [0,1]$

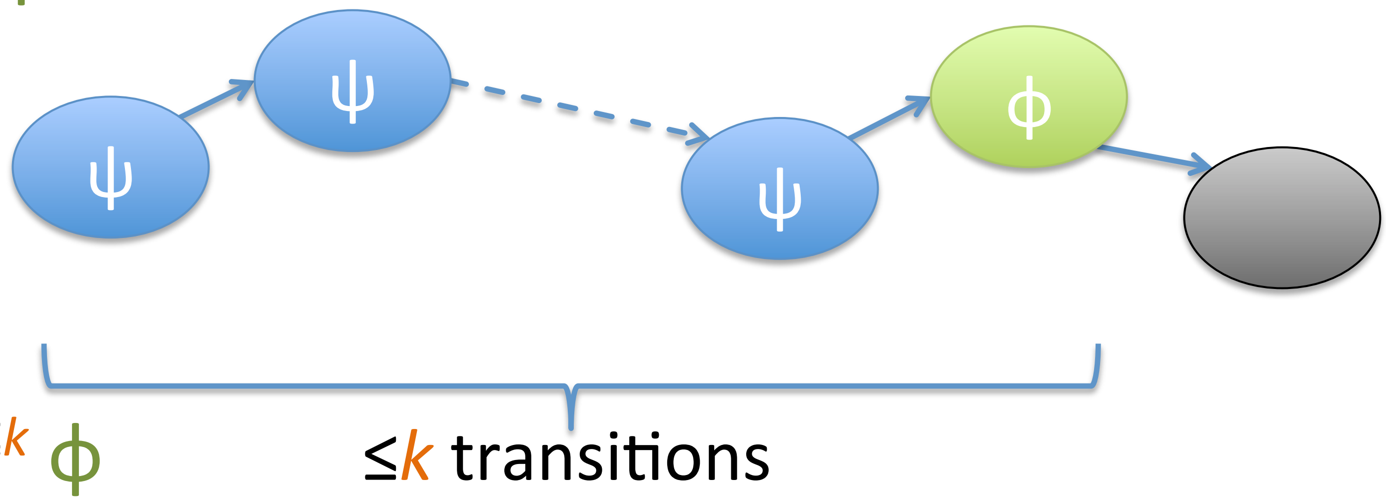
- path formulas Π
  - $X\ \phi$ — next state
  - $\psi\ U\ \phi$ — unbounded until
  - $\psi\ U^{\leq k}\ \phi$ — bounded until

Path formulas

- X ϕ



- ψ U ϕ



- $\psi\ U^{\leq k}\ \phi$

Example formulas...

- simple communication protocol:
  sender sends a message;
  it gets through with 90% probability;
  otherwise it is resent until success.
  - Model this Markov chain.
  - Formulate a useful PCTL property for this MC.
- Craps:
  - Formulate a useful PCTL property for Craps.

PCTL semantics

- Formal definition:
- When does a state satisfy a state formula?
- When does a path satisfy a path formula?

Formal definition: labelled DTMC

- A Markov chain consists of:
  - S — finite set of states (often $S = \{1, 2, \ldots n\}$)
  - $\mathbf{P}: S \times S \rightarrow [0,1]$ — transition probability matrix (with row sums = 1)
  - $\pi_0: S \rightarrow [0,1]$ — initial state distribution (sometimes)
  - $L: S \rightarrow 2^{AP}$ — labelling with atomic propositions

PCTL semantics: state formulas

Sat(ϕ) = set of states that satisfy ϕ:

- Sat(a) = $\{s \mid a \in L(s)\}$
- Sat($\neg\phi$) = $S \setminus$ Sat(ϕ)
- Sat($\phi \vee \psi$) = Sat(ϕ) $\cup$ Sat(ψ)

- Sat($\mathbf{P}_{\geq p}(\Pi)$) = $\{s \mid \text{Prob}_s\ \text{Sat}(\Pi) \geq p\}$ etc.

- Prob_s Sat(Π) = probability to get a Π-path, if you start in s

PCTL semantics: path formulas

Sat(Π) = set of paths that satisfy Π:

- Sat(X ϕ) = $\{\sigma \mid \sigma[1] \in \text{Sat}(\phi)\}$
- Sat(ψ U ϕ) = $\{\sigma \mid \exists\, i \geq 0, \sigma[i] \in \text{Sat}(\phi)$ and $\forall\, j < i, \sigma[j] \in \text{Sat}(\psi)\}$
- Sat(ψ $U^{\leq k}$ ϕ) = $\{\sigma \mid \exists\, i \geq 0, i \leq k$ and $\sigma[i] \in \text{Sat}(\phi)$ and $\forall\, j < i, \sigma[j] \in \text{Sat}(\psi)\}$

Measurability

- For every PCTL path formula,
  the set of paths satisfying the formula
  is measurable (i. e. is in the σ-algebra).

- Proof: one can construct the set
  as a (disjoint) union of cylinder sets...
  (details on board)

PCTL model checking

- construct satisfaction sets bottom-up
  (as with CTL model checking)

- new step: probabilistic operator $\mathbf{P}_{\geq p}(\Pi)$
  → constrained probabilistic reachability

Example...

- Craps, formula $\mathbf{P}_{>0.3}(\neg \mathit{state}_4 \ U^{\leq 6} \ \mathit{win})$
- “Is the probability to win with up to 6 rolls > 0.3, but without the first roll being 4?”

- Need to calculate the probability of Sat($\neg \mathit{state}_4 \ U^{\leq 6} \ \mathit{win}$) = union of cylinder sets... (on the blackboard)

How to calculate the probability of Sat($\neg$*lose* $U^{\leq k}$ *win*)

- Make all states in Sat(*win*) absorbing.
  Reason: All paths reaching a *win*-state satisfy the formula.
- In the modified MC,
  Sat($\neg$*lose* $U^{\leq k}$ *win*) = Sat($\neg$*lose* $U^{=k}$ *win*)

- Make all states in Sat(*lose*) absorbing.
  Reason: All paths reaching a *lose*-state falsify the formula.
- In the modified MC,
  Sat($\neg$*lose* $U^{=k}$ *win*) = Sat(*true* $U^{=k}$ *win*)

How to calculate the probability of Sat($\color{red}{true}$ $U^{=k}$ *win*)

- Sat(*true* $U^{=k}$ *win*) = set of paths that reach *win* in *k* steps
- Transient probability for m… chain: Pr(Sat(*true* $U^{=k}$ *win*)) = $\pi_k = \pi_0 \cdot P^k$

  $\pi_k(s)$ = probability to be in state *s* after *k* steps
- sum over all *win*-states: $\Sigma_{s \in \mathrm{Sat}(win)} \pi_k(s)$
- probability = $\pi_0 \cdot P^k \cdot \mathbf{1}_{win}$

  vector: 1 in *win*-states, 0 in other states
- calculate this for all initial states

  $\pi_0 = (1, 0, 0, \ldots)$ $\pi_0 = (0, 1, 0, 0, \ldots)$ etc.

  → vector $P^k \cdot \mathbf{1}_{win}$

How to calculate $\mathrm{Sat}(\mathbf{P}_{\geq p}(\neg \mathit{lose}\ \mathrm{U}^{\leq k}\ \mathit{win}))$

- modify Markov chain:
  make all *win*- and all *lose*-states absorbing
- calculate $\pi_{win} = P^{k} \cdot \mathbf{1}_{win}$
- For every state *s*:
  If $\pi_{win}(s) \geq p$, then $s \in \mathrm{Sat}(P_{\geq p}(\neg \mathit{lose}\ \mathrm{U}^{\leq k}\ \mathit{win}))$

$\pi_{win}(s)$ = probability that $(\neg \mathit{lose}\ \mathrm{U}^{\leq k}\ \mathit{win})$, if starting in *s*

- Advantage: calculate for each state *s* at once!

Example...

- calculate the probabilities for the Craps example and $\mathbf{P}_{>0.3}(\neg(\mathit{state}\ 4)\ \mathrm{U}^{\leq 6}\ \mathit{win})$

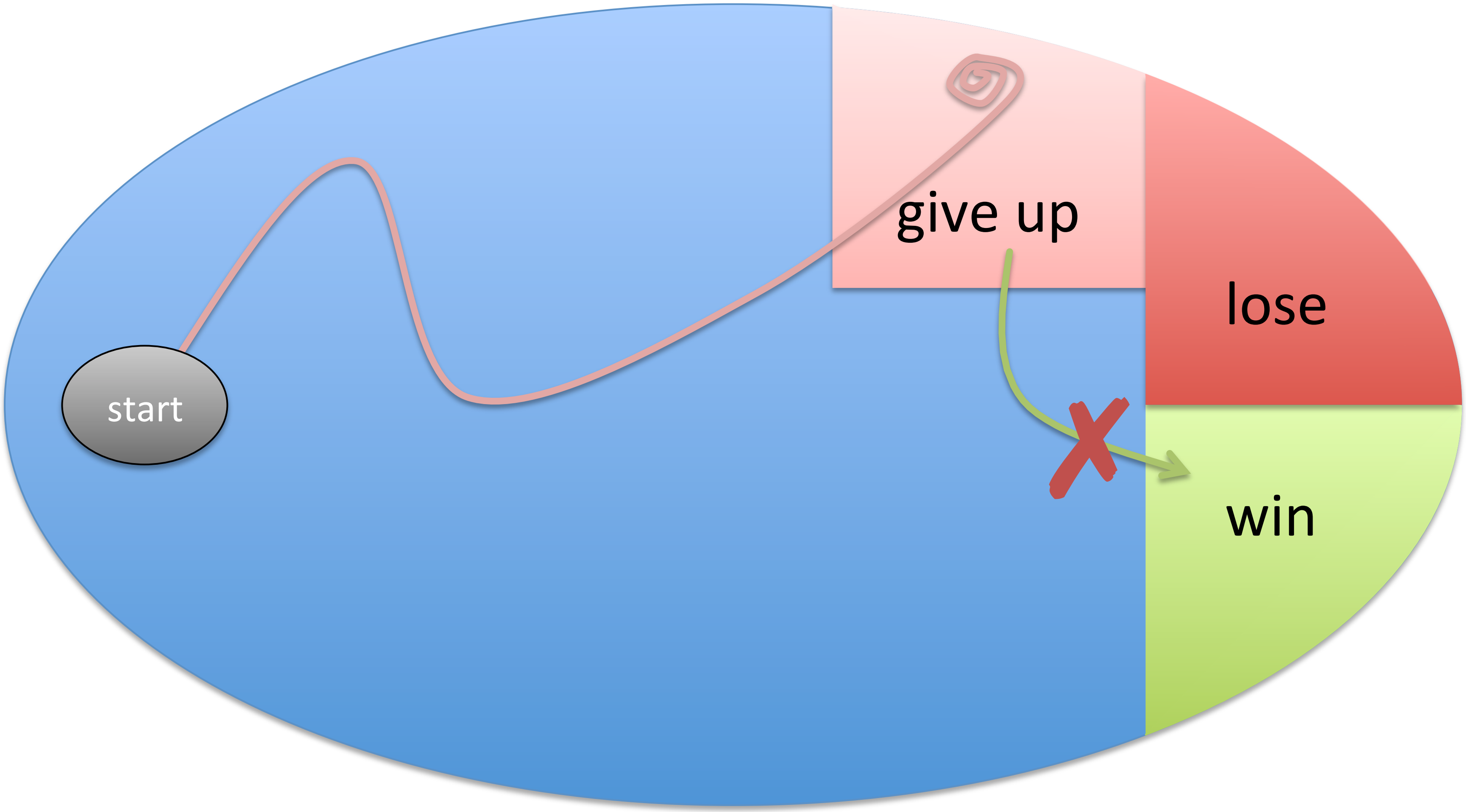
- Be careful: “lose” gets a new meaning here.

Optimization

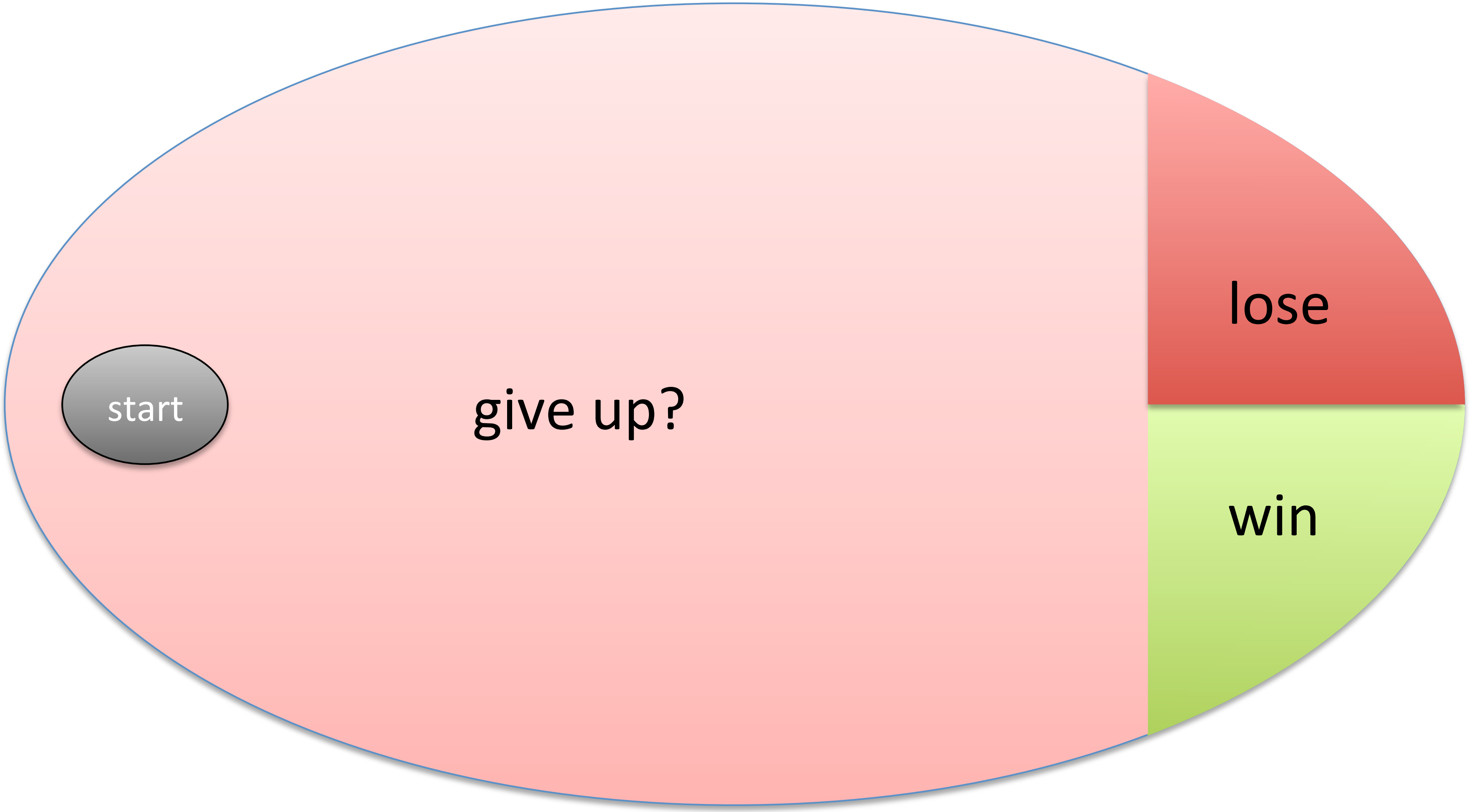
- There may be stalemate states
  (Poland is not yet lost, but it will never win)
- In these states, probability to reach *win* is = 0
- Finding stalemate states is simpler
  than matrix calculations

- Therefore:
  also make stalemate states absorbing.

Stalemate states



How to find stalemate states



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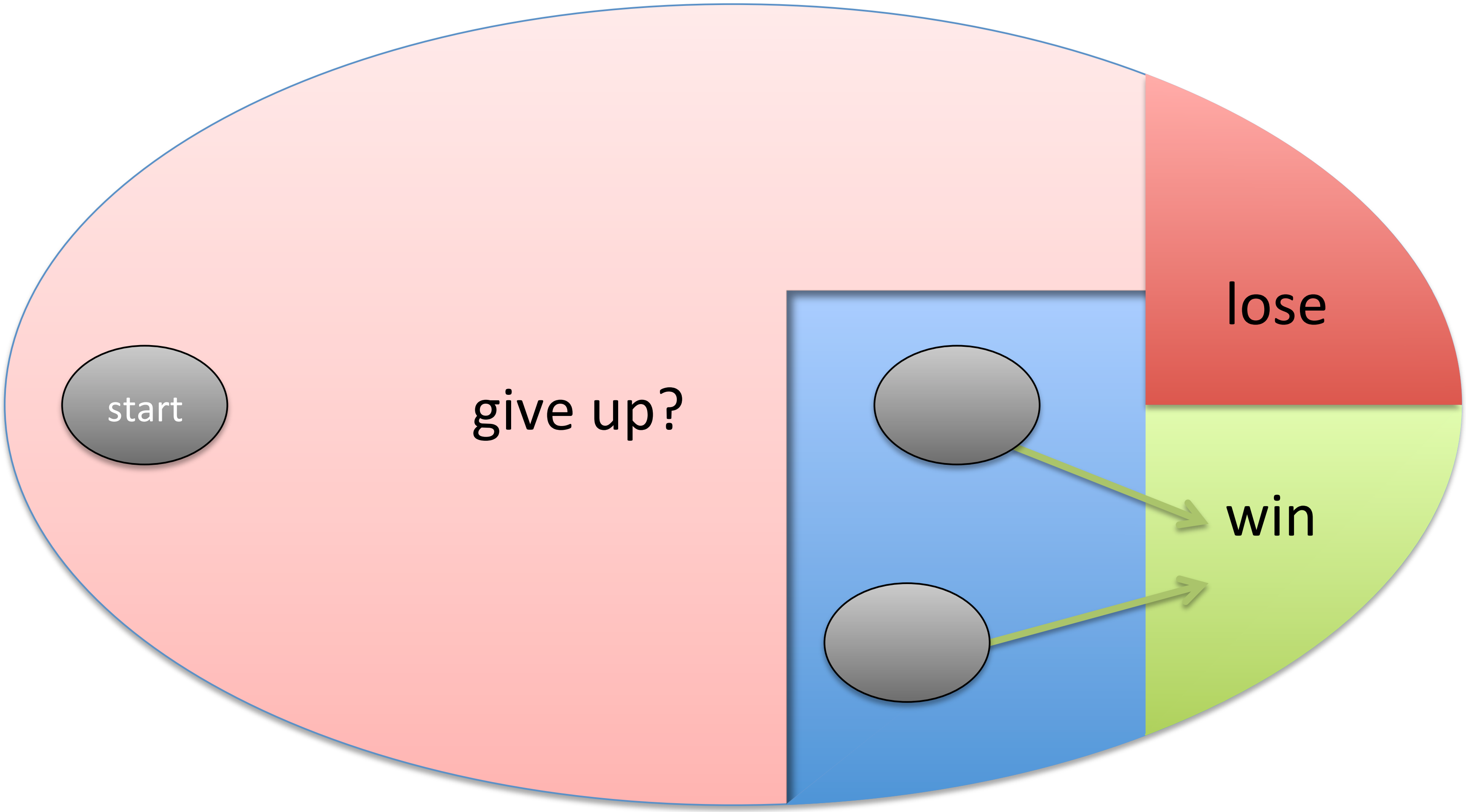

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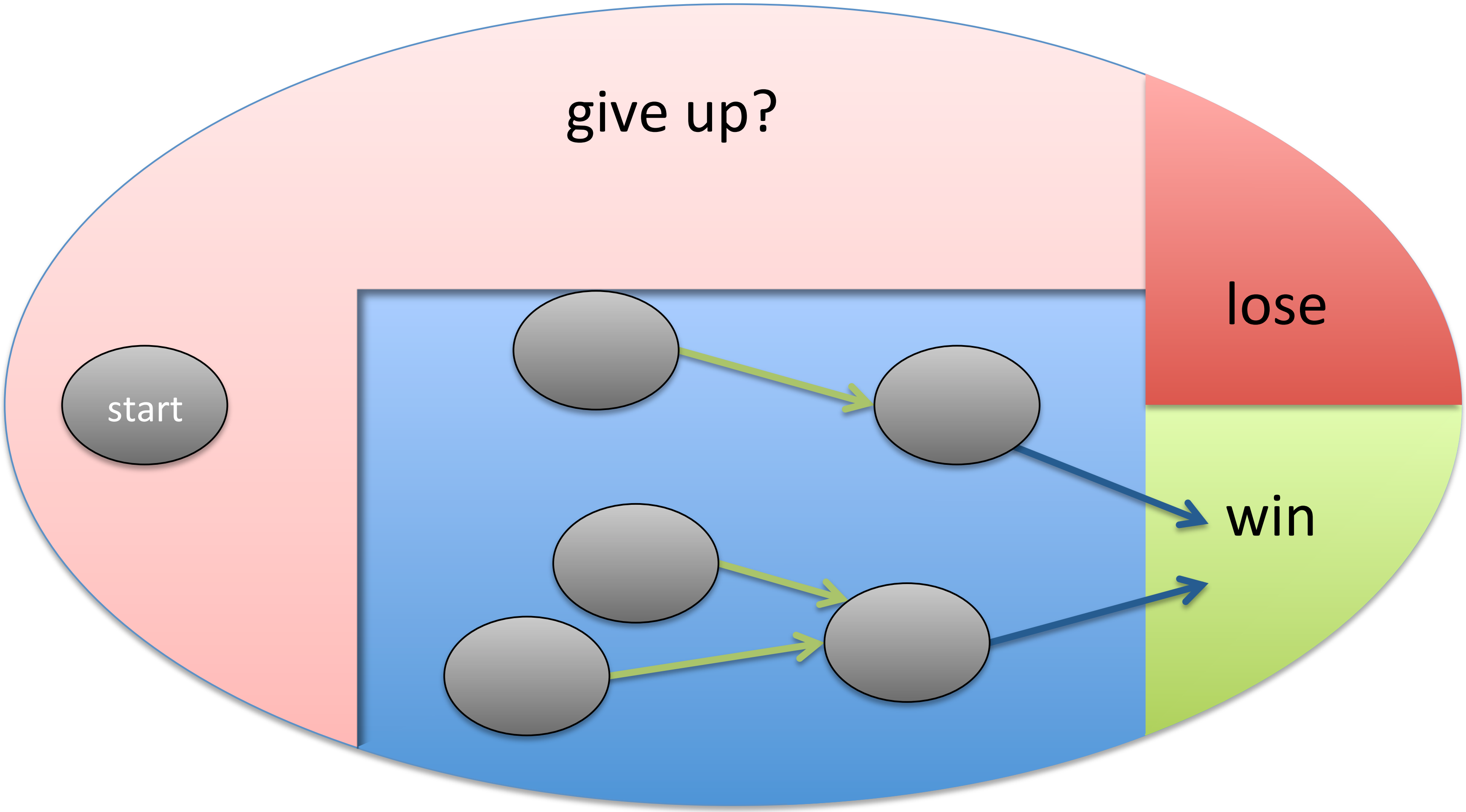

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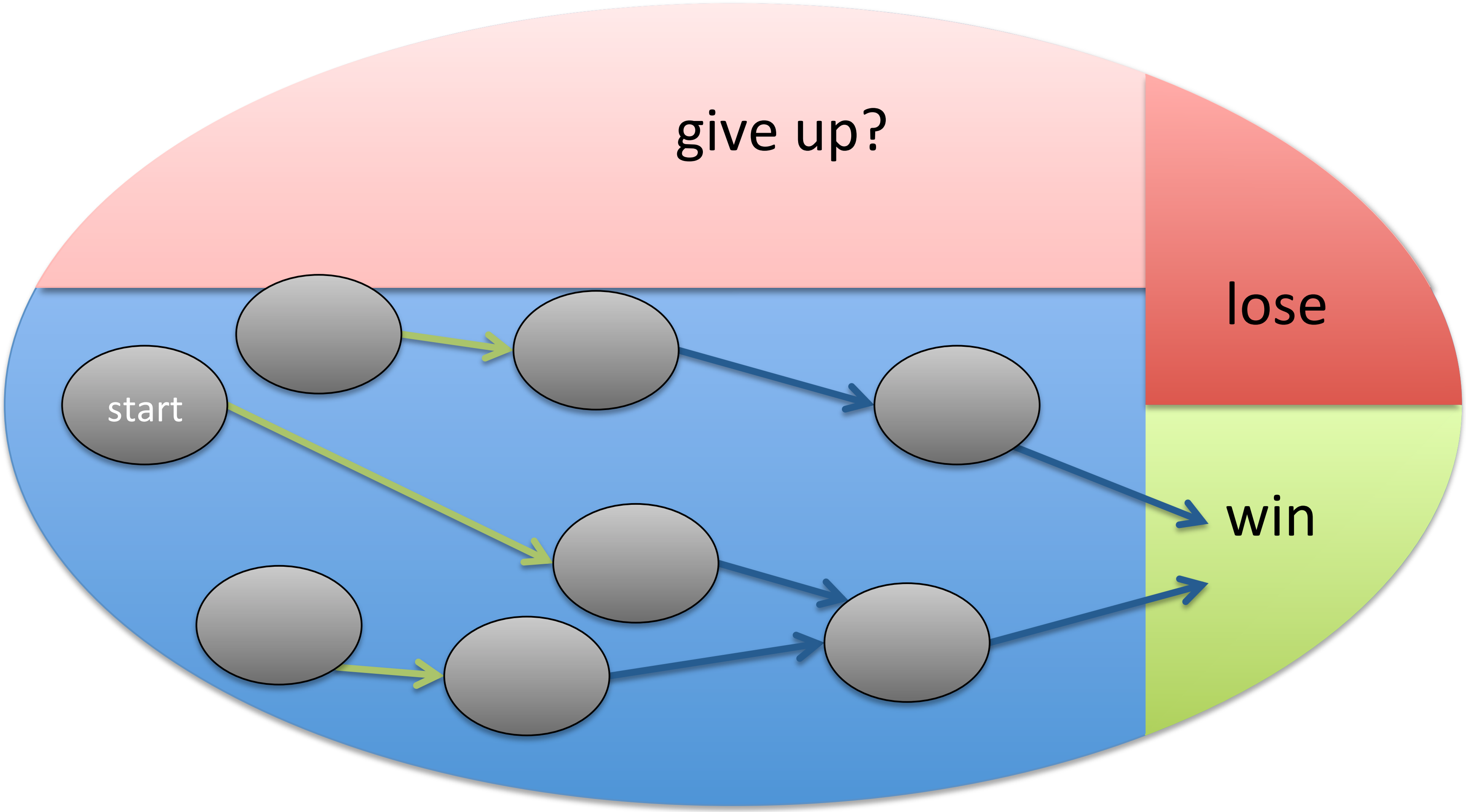

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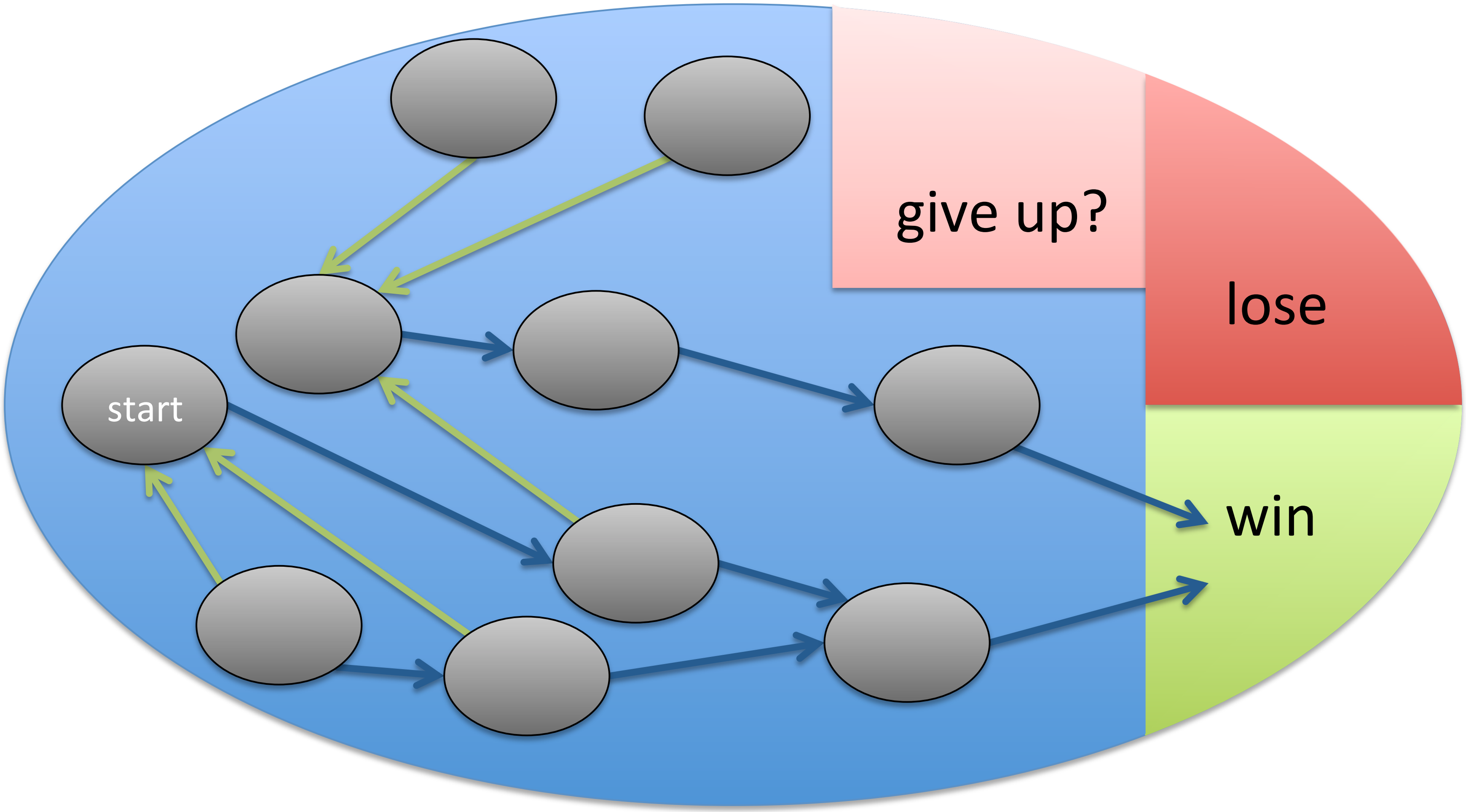

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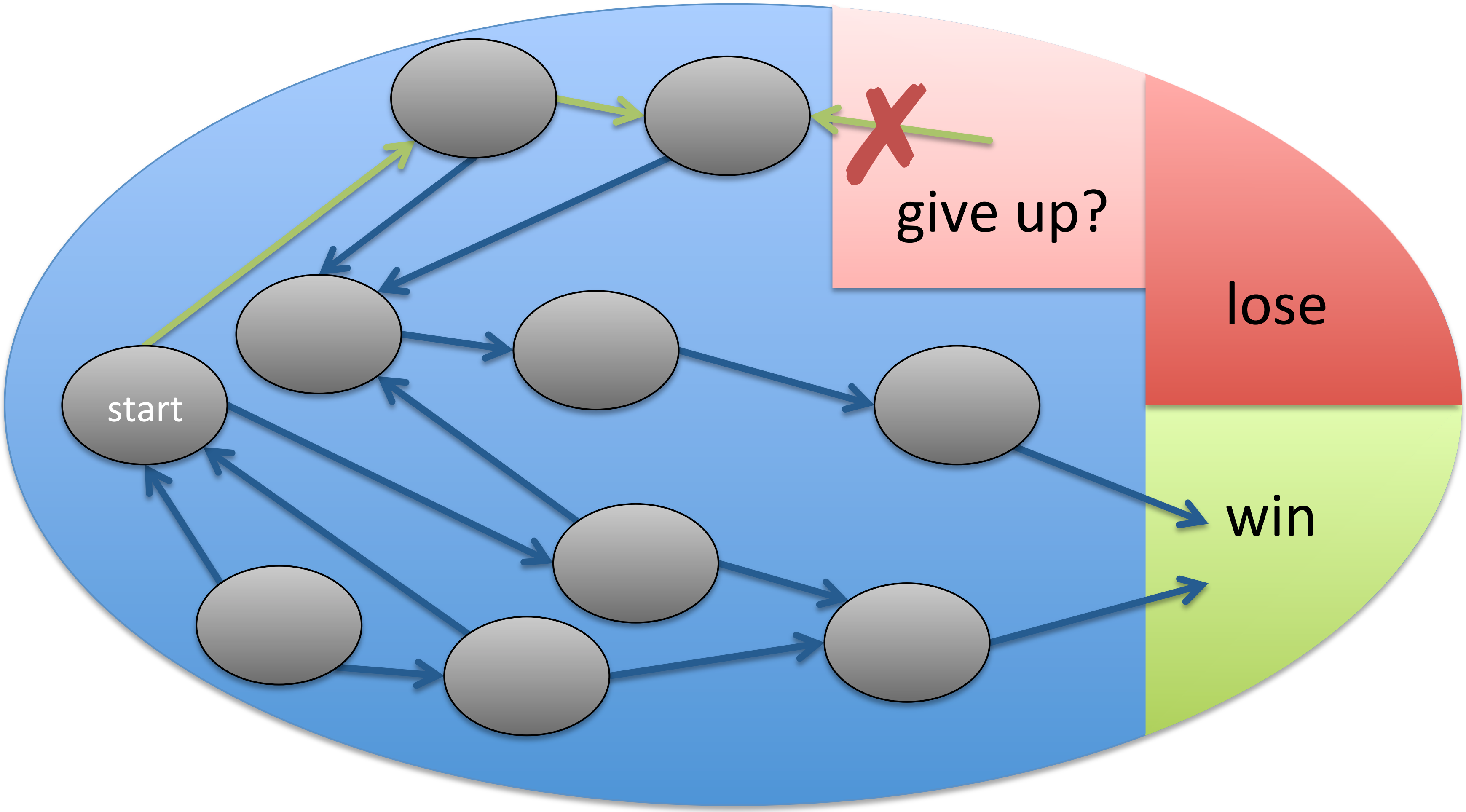
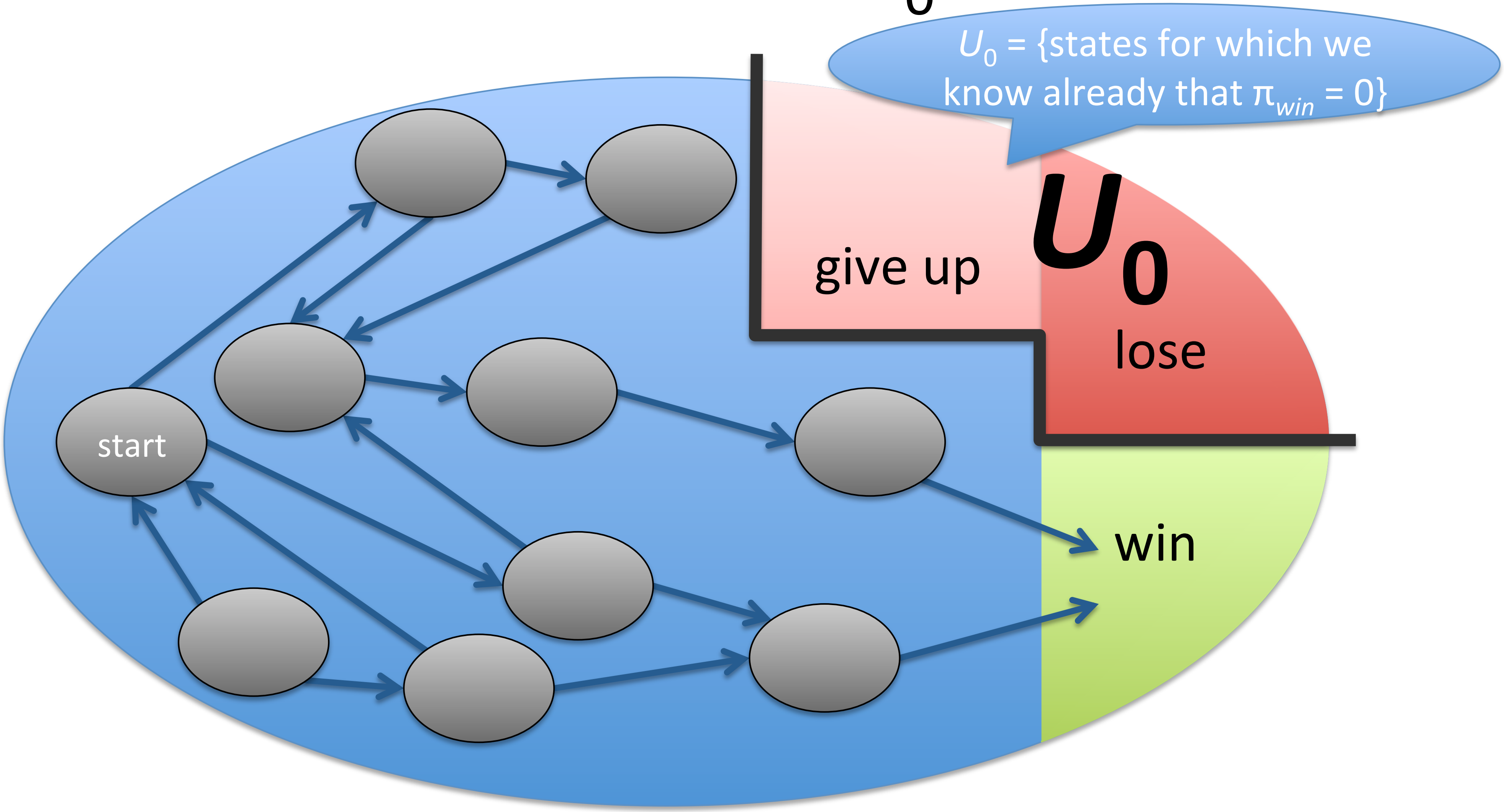


How to find U_0



How to find U_0

- create list of winning states
- iterate through list from start to end:
  for each state,
  add non-losing predecessors to end of list
- when iteration is complete,
  the list contains all states except U_0

Recapitulation

- PCTL: a logic to describe properties of Markov chains
- most important property: until formula = constrained probabilistic reachability
- compute probability with equation system
  - solution is unique if ...