Quantitative Logics
CTMC exercises

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1. Prove the following property of a CTMC: If a state s_0 has two transitions, say $R(s_0, s_1) = \lambda$ and $R(s_0, s_2) = \mu$, then state s_1 is reached with probability $\lambda/(\lambda+\mu)$ and state s_2 is reached with probability $\mu/(\lambda+\mu)$. (Hint: Look at the joint pdf for the two events, $f(x_1, x_2)$ = the probability density that the transition to s_1 gets enabled at time x_1 and the transition to s_2 gets enabled at time time x_2, $= f_\lambda(x_1) \cdot f_\mu(x_2)$. Integrate it over the relevant part of the parameters.)

2. Radioactive decay is memoryless: given a radioactive atom, the time until its decay is a random variable that does not depend on how old the atom is already. Therefore, radioactive decay can be described by a probability distribution, e. g. $F_\alpha(t)$ = the probability that a uranium atom has decayed before time t, for a suitable α.

   (a) Find the relation between the rate α of an exponential distribution and the half life. (Hint: The half life differs from the expectation of F_α.)

   (b) Draw a continuous time Markov chain for the decay of a ^{235}U atom based on the following information.

Element	Isotope	Half life	Decay product(s)
Actinium	^{227}Ac	21.77 a	98.62% ^{227}Th 1.38% ^{223}Fr
Astatine	^{215}At	0.1 ms	^{211}Bi
Bismuth	^{211}Bi	2.13 min	99.72% ^{207}Tl 0.28% ^{211}Po
Francium	^{223}Fr	21.8 min	^{223}Ra
Lead	^{207}Pb ^{211}Pb	stable 36.1 min	— ^{211}Bi
Polonium	^{211}Po ^{215}Po	0.516 s 1.78 ms	^{207}Pb 99.99977% ^{211}Pb 0.00023% ^{215}At
Protactinium	^{231}Pa	32500 a	^{227}Ac
Radium	^{223}Ra	11.4 d	^{219}Rn
Radon	^{219}Rn	4.0 s	^{215}Po
Thallium	^{207}Tl	4.77 min	^{207}Pb
Thorium	^{227}Th ^{231}Th	18.5 d 25.5 h	^{223}Ra ^{231}Pa
Uranium	^{235}U	$7.04 \cdot 10^8$ a	^{231}Th

3. Prove that the set of all paths that reach goal state s_{goal} within time at most t_{goal} is measurable, i. e. is the union of cylinder sets. (Note that the intervals in the cylinder sets restrict the sojourn time in a single state, not the total time before reaching the goal state. For cylinders containing intermediary states, approximate the time that they pass from one state to the next and take the limit.)